Geometric focusing of internal waves – A linear theory

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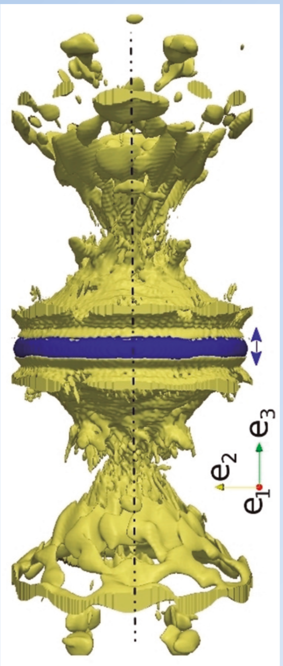
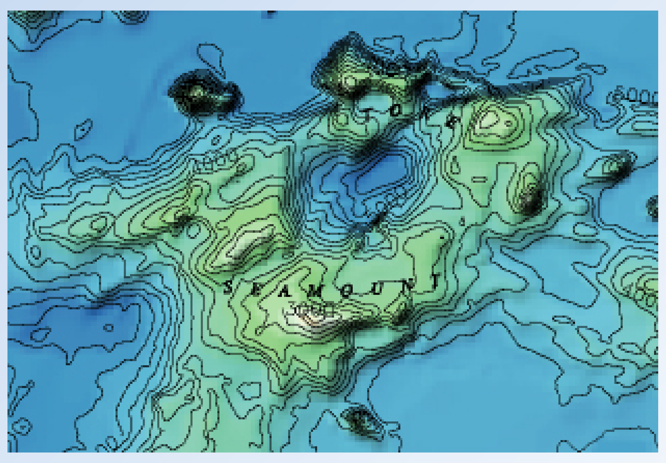
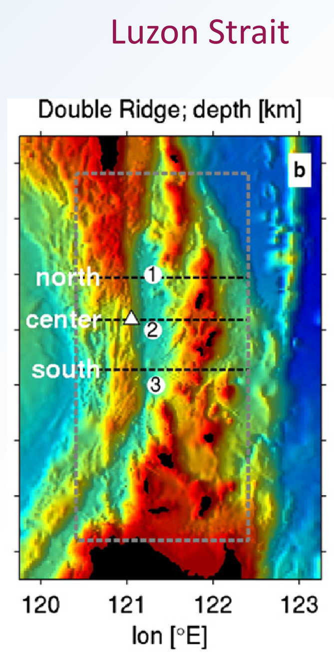
Motivations



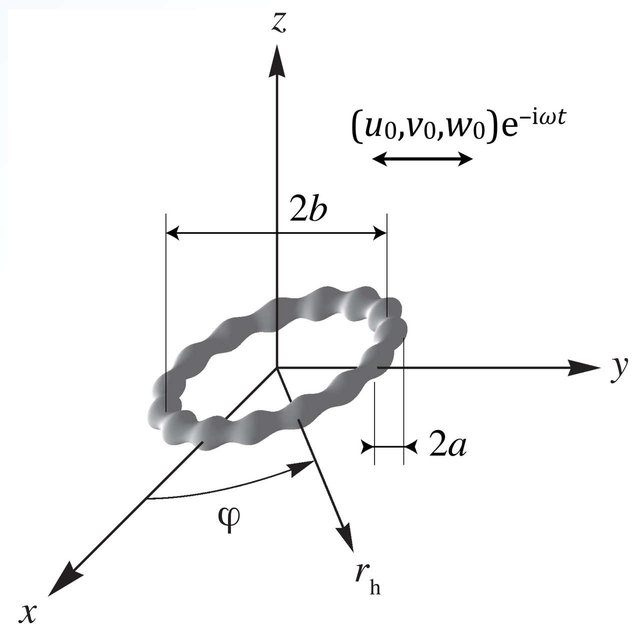




- A 3D mechanism of energy concentration
- Operating in the fluid interior
- Oceanic ridges are often curved

(Duran-Matute et al. *PRE* 2014)

Theoretical outline



A horizontal circular annulus oscillates in a stratified fluid of frequency N and viscosity ν

- Generation parameters

$$\epsilon = \frac{b}{a},\quad \Omega = \frac{\omega}{N},\quad St = \frac{\omega a^2}{\nu},\quad Ke = \frac{|\boldsymbol{u}_0|}{\omega a}$$

- Propagation parameters

$$\theta = \arccos\left(\frac{\omega}{N}\right),\quad \beta = \frac{\nu}{2\omega\tan\theta}$$

Each cross-section admits a 2D representation as a source of mass

$$q(x,z) \equiv \nabla\cdot\boldsymbol{u} = u_0\frac{\partial}{\partial x}f(x,z) + w_0\frac{\partial}{\partial z}g(x,z)$$

At large aspect ratio (ε » 1), this representation remains valid for the annulus

$$q(x,y,z) = \left(u_0\frac{\partial}{\partial x} + v_0\frac{\partial}{\partial y}\right)f(r_\mathrm{h}-b,z;\varphi) + w_0\frac{\partial}{\partial z}g(r_\mathrm{h}-b,z;\varphi)$$

Separate expansions at the global scale b of the annulus and local scale a of the cross-sections lead to a uniform expression of the waves in terms of the Fourier components of the forcing

$$f_n^{(c,s)}(k,m) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}z\int_{-\pi}^{\pi}\mathrm{d}\varphi\, f(x,z;\varphi)(\cos,\sin)(kx)\exp(-\mathrm{i}mz)\mathrm{e}^{-\mathrm{i}n\varphi}$$

The vertical velocity follows as (and similarly for the other components)

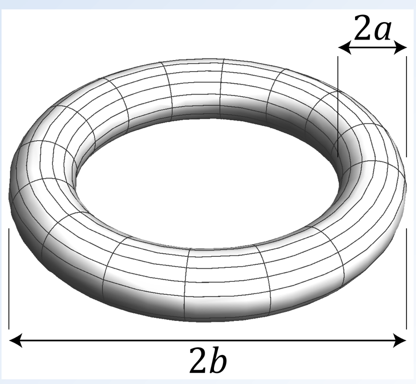
$$w = \frac{b}{4}\cos^2\theta\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{sign}\,z\sum_{n=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}n\varphi}\int_0^{\infty}\mathrm{d}\kappa\,\kappa^2 A_n(\kappa)\exp\left(-\frac{\beta\kappa^3|z|}{\cos\theta}\right)J_n(\kappa r_\mathrm{h}\cos\theta)\exp(-\mathrm{i}\kappa|z|\sin\theta)$$

with

$$A_n(\kappa) = -2\mathrm{i}w_0\sin\theta\,\mathrm{sign}\,z\left[J_n(\kappa b\cos\theta)g_n^{(c)} - Y_n(\kappa b\cos\theta)g_n^{(s)}\right](\kappa\cos\theta,-\kappa\sin\theta\,\mathrm{sign}\,z)$$
$$+\sum_{\pm}(\mathrm{i}v_0 \pm u_0)\cos\theta\left[J_{n\pm1}(\kappa b\cos\theta)f_{n\pm1}^{(c)} - Y_{n\pm1}(\kappa b\cos\theta)f_{n\pm1}^{(s)}\right](\kappa\cos\theta,-\kappa\sin\theta\,\mathrm{sign}\,z)$$

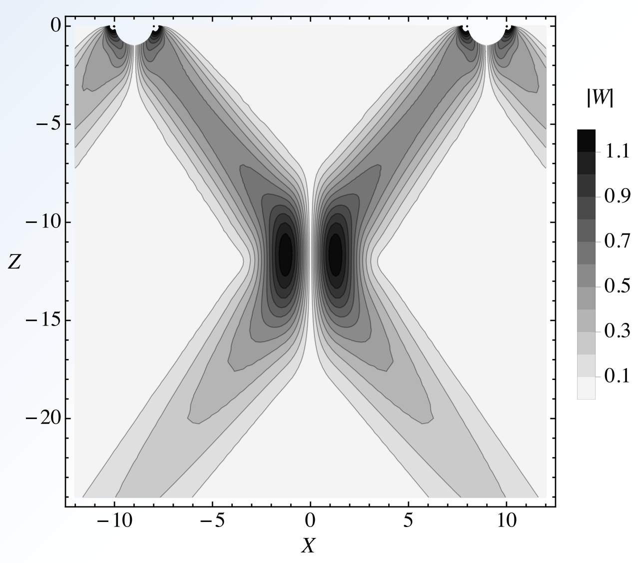
Axisymmetric torus

$$q(x,z) = (1+\mathrm{i}\tan\theta)u_0\frac{x}{a}\delta\left(\sqrt{x^2+z^2}-a\right)$$

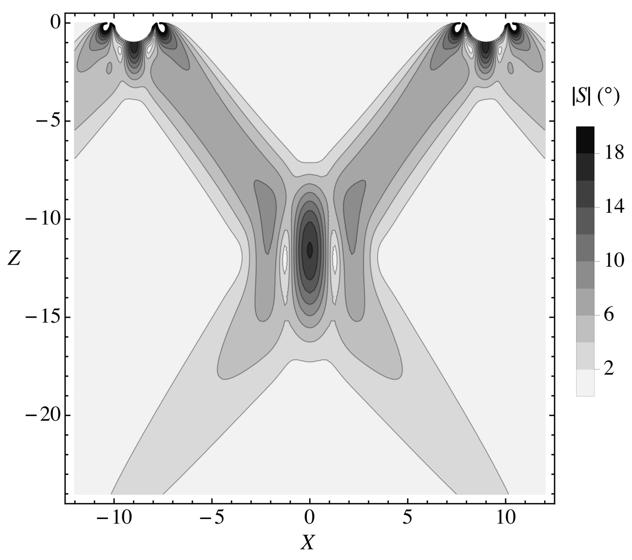


$\varepsilon = 9,\ \Omega = 0.8$

$St = 120,\ Ke = 0.2$



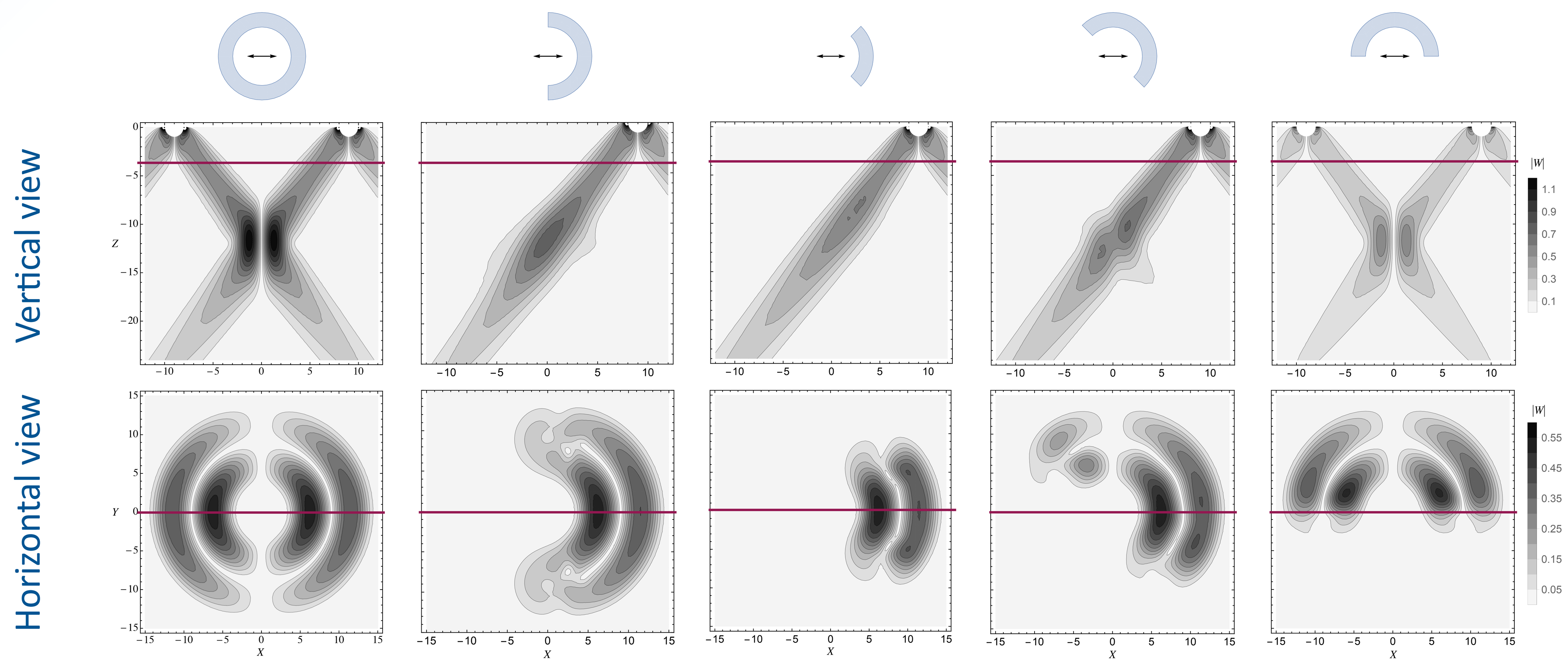
Vertical velocity



Isopycnic slope

Focusing yields significant wave slope even at low oscillation amplitude Ke

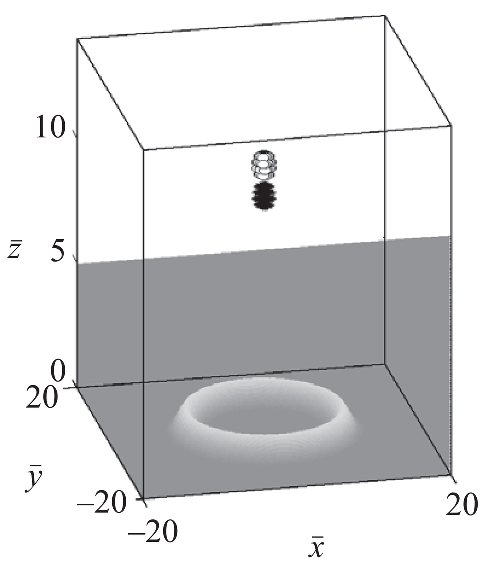
Non-axisymmetric torus



Focusing remains present for partial or oblique annuli

Gaussian ring

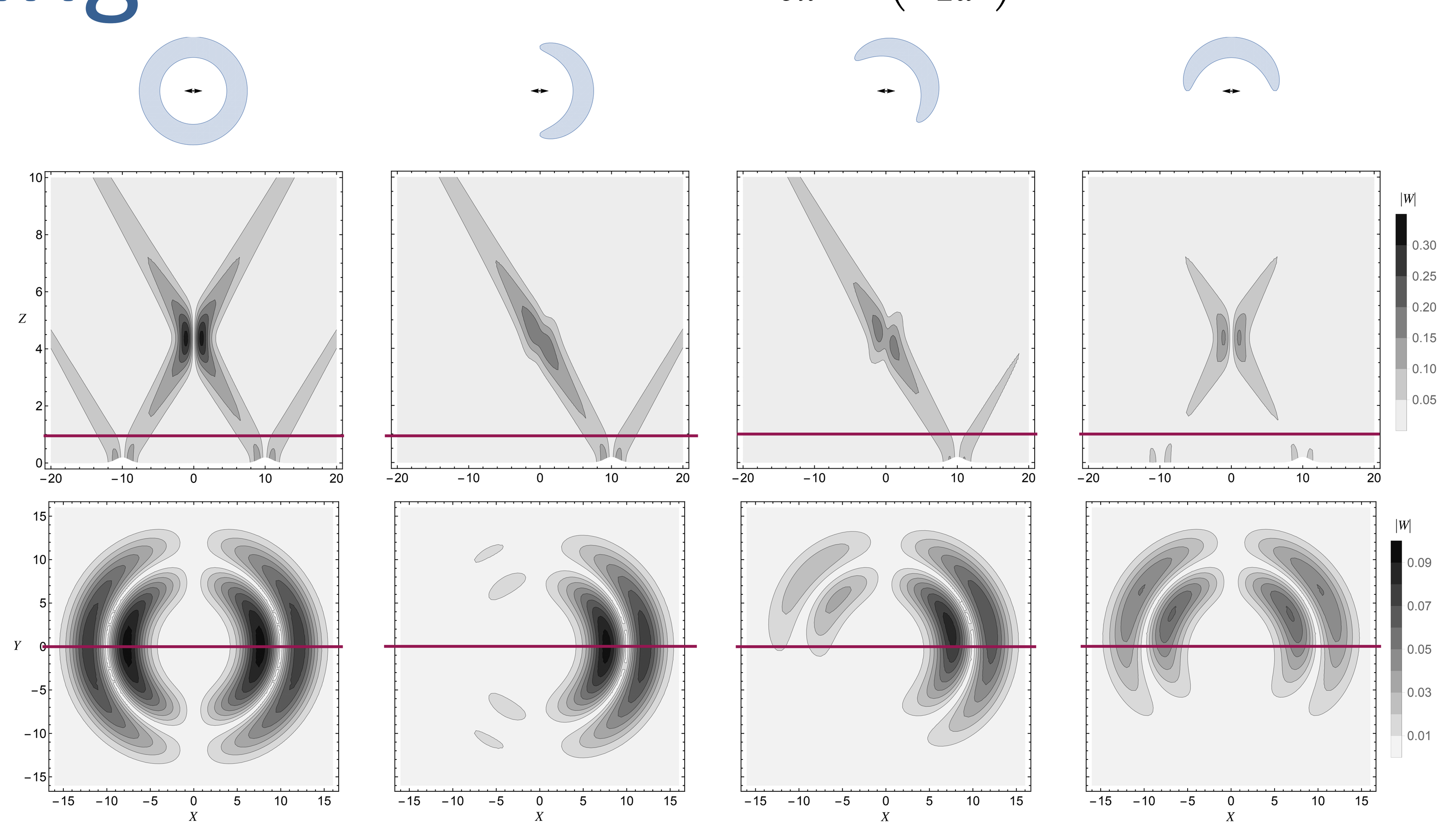
$$q(x,z) = 2u_0 h\delta(z)\frac{\partial}{\partial x}\exp\left(-\frac{x^2}{2a^2}\right)$$



(Bühler & Muller *JFM* 2007; Grisouard & Bühler *JFM* 2012)

$\varepsilon = 10,\ h/a = 0.2,\ \Omega = 0.4$

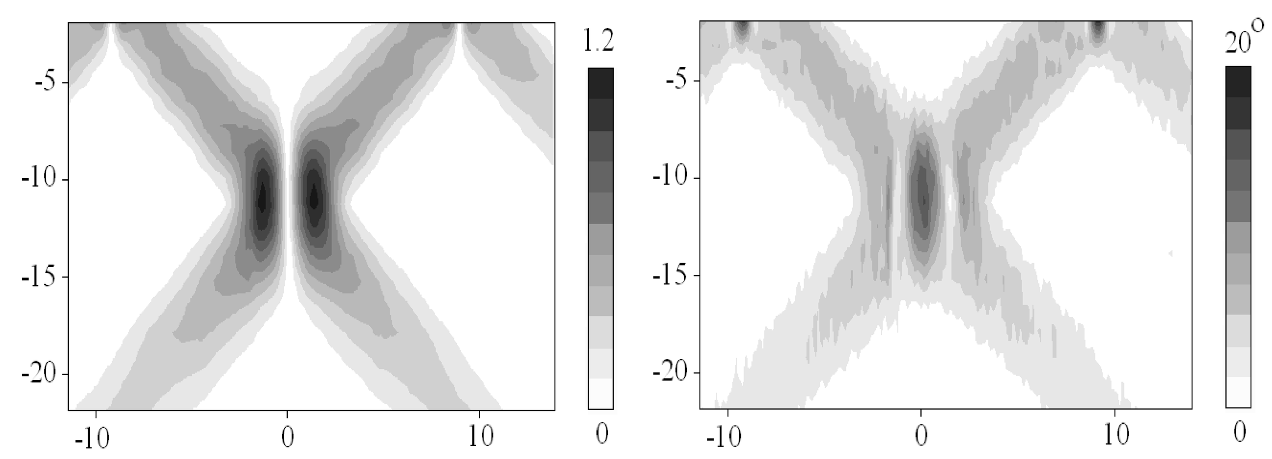
$St = \infty,\ Ke = 0.04$



Focusing remains present for subcritical forcing

Perspectives

- Focusing results from horizontal curvature, making it robust to lacks of axisymmetry or criticality
- Theory for the full torus is confirmed by experiment



Vertical velocity | Isopycnic slope

(Ermanyuk, Shmakova & Flór *submitted to JFM*)

- To explore higher St leading to bimodal waves with smaller structures and higher slopes, we move on to the Coriolis platform