A statistical mechanics approach of mixing in stratified fluids

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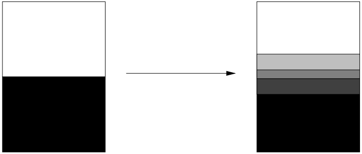
Density Stratification, Turbulence, but How Much Mixing?



Ivey Winters Koseff 2008

- **Global Richardson or Froude number**

$$Ri = \frac{H\Delta b}{U^2} = \frac{1}{Fr^2}$$

- **Buoyancy field**

$$b(\mathbf{x}, t) = g\frac{\varrho_0 - \varrho(\mathbf{x}, t)}{\varrho_0}$$

- **Objective: a model for** $\overline{b}(z, t)$
- **Here: a statistical mechanics approach**

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TURBULENCE IN STRATIFIED FLUIDS: A REVIEW

E. J. Hopfinger

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Turbulence in stratified fluids: a Review, Hopfinger JGR 1987

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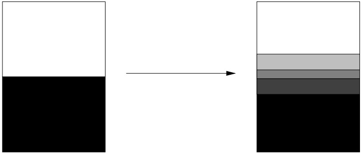

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Outline

- Equilibrium theory for $\overline{b}(z)$
- Application: computation of mixing efficiency
- An out-of-equilibrium model for $\overline{b}(z, t)$
- Application: entrainment across a density interface

Freely evolving Boussinesq fluid

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} = -\nabla\phi + b\mathbf{e}_z \quad (1)$$

$$\partial_t b + (\mathbf{u}\cdot\nabla)\,b = 0 \quad (2)$$

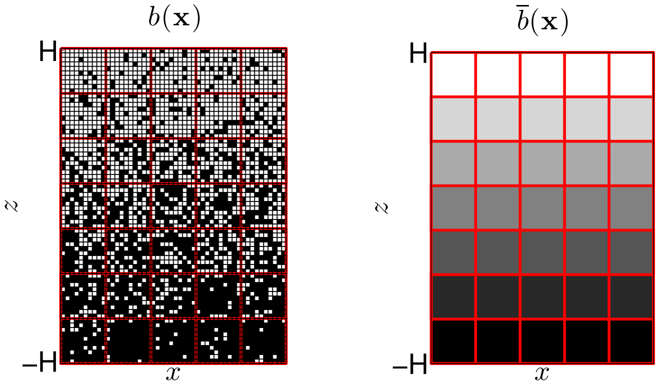
$$\nabla\cdot\mathbf{u} = 0 \quad (3)$$

Dynamical invariants: global distribution of buoyancy levels and total energy if the velocity remains sufficiently smooth.

$$E = \int \left(\frac{1}{2}\mathbf{u}^2 - bz\right) d^3\mathbf{x}$$

Equilibrium statistical mechanics

Microstate: any field $b(\mathbf{x}), \mathbf{u}(\mathbf{x})$ satisfying dynamical constraints
Macrostate: $\rho(\sigma, \mathbf{v}, \mathbf{x})$ frequency of occurrence of $b = \sigma$ and $\mathbf{u} = \mathbf{v}$ in the vicinity of $\mathbf{x}$.
Coarse-graining procedure: $\overline{b^i}(\mathbf{x}) = \int \rho \sigma^i \mathrm{d}\sigma \mathrm{d}\mathbf{v}$



Similar to Robert-Sommeria-Miller statistical mechanics of 2D turbulence.

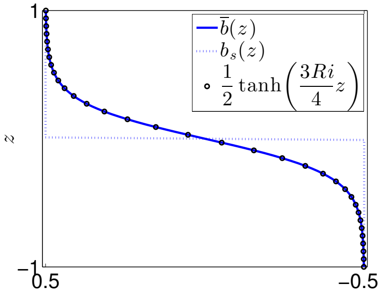
Properties of the equilibria

$$
\begin{aligned}
\rho &= N e^{-\beta \frac{\mathbf{v}^2}{2}} \rho_b(\sigma, z) \\
\rho_b &= \frac{e^{\beta \sigma z + \gamma(\sigma)}}{\mathcal{Z}(z)} \\
\beta &= \frac{3}{2 e_c}
\end{aligned}
\tag{4}
$$

- **Velocity and buoyancy distributions are independent**
- The **buoyancy distribution** depends only on height z.
- The **velocity distribution is gaussian** and spatially homogeneous, with kinetic energy e_c.
- Local **buoyancy fluctuations** are also proportional to e_c.

Two level configuration

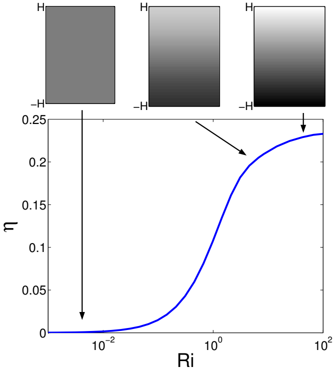
Buoyancy distribution with two levels $\pm\Delta b/2$ in equal proportion:



$$Ri = \frac{\Delta b H}{e_c}.$$

- $Ri \ll 1$ the buoyancy field is fully homogenized
- $Ri \gg 1$ the buoyancy field is "sorted".

Mixing efficiency



Input: $b_s(z), E_{inj}$.

Assume the system evenly explores phase space

Output: $\overline{b}(z)$

$$\eta \equiv \frac{E_p[\overline{b}] - E_p[b_s]}{E_{inj}}, \qquad Ri = \frac{\Delta b H}{e_c}$$

- **Complete mixing for** $Ri \ll 1$: $\eta \propto Ri$
- **Energy equipartition for** $Ri \gg 1$: $\eta = \frac{1}{4}$
- Confirms and generalises McEwan JFM 1983.

Relaxation equations

$$\partial_t \rho_b = \partial_z \left[D \left(\partial_z \rho_b - \frac{3}{2e_c} \left(\sigma - \overline{b} \right) \rho_b \right) \right] .$$

- **Converges towards the equilibrium state.**
- **Satisfies basic conservation laws of the original system.**
- **Model turbulent transport and restratification.**

Dynamical model

$$\partial_t \rho_b = \partial_z \left[D \left(\partial_z \rho_b - \frac{3}{2e_c} \left(\sigma - \overline{b} \right) \rho_b \right) \right] + \frac{e_c^{1/2}}{\Lambda_b} \mathcal{D}_{mix}[\rho_b] \, .$$

- **Dissipation operator** $\mathcal{D}_{mix}$ conserves the norm and the mean value of ρ_b at each height z.
- **Turbulent diffusion** $D = l_b e_c^{1/2}$.
- **Dynamical equation for** $e_c(z, t)$.

In the following, we assume to simplify $e_c = cst$, $l_b = cst$, $\Lambda_b = cst$.

Application to a forced-dissipated configuration

Buoyancy prescribed at upper and lower boundaries

$$b(H) = \frac{1}{2}, \quad b(-H) = -\frac{1}{2}$$

Parameters

$$l_b^* = \frac{l_b}{H}, \quad Ri = \frac{H \Delta b}{e_c}, \quad \epsilon = \frac{H^2}{\Lambda_b l_b}$$

$$l_b^* Ri^{1/2} \partial_t \rho_b = \partial_z \left[\partial_z \rho_b - \frac{3}{2} Ri \left(\sigma - \overline{b} \right) \rho_b \right] + \epsilon \mathcal{D}_{mix}[\rho_b] \ .$$

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Formation of thermoclines in zero-mean-shear turbulence subjected to a stabilizing buoyancy flux

By E. J. HOPFINGER† AND P. F. LINDEN
Department of Applied Mathematics and Theoretical Physics, University of Cambridge

(Received 13 February 1981)

Parameters

$$l_b^* = \frac{l_b}{H}, \qquad Ri = \frac{H\Delta b}{e_c}, \qquad \epsilon = \frac{H^2}{\Lambda_b l_b}$$

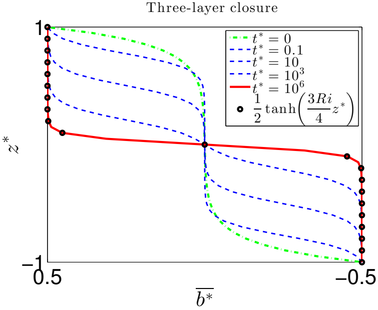
$$l_b^* Ri^{1/2} \partial_t \rho_b = \partial_z \left[\partial_z \rho_b - \frac{3}{2} Ri \left(\sigma - \overline{b} \right) \rho_b \right] + \epsilon \mathcal{D}_{mix}[\rho_b] \, .$$

Closure with 3 buoyancy levels

$$b \in \left\{-\frac{1}{2}, 0, +\frac{1}{2}\right\}, \quad \overline{b} = \frac{1}{2}(p_+ - p_-)$$

$$I_b^* Ri^{1/2} \partial_t p_+ = \partial_{zz} p_+ + \frac{3}{2} Ri \partial_z \left[\left(\frac{1}{2} - \overline{b}\right) p_+\right] - \epsilon p_+ p_- .$$

$$I_b^* Ri^{1/2} \partial_t p_- = \partial_{zz} p_- - \frac{3}{2} Ri \partial_z \left[\left(\frac{1}{2} + \overline{b}\right) p_-\right] - \epsilon p_+ p_- .$$



$Ri \gg 1, \quad \epsilon \ll 1$

Spontaneous emergence of a sharp interface

Note: a Gaussian closure for ρ_b leads to [Mellor Yamada 1978]: no interface.

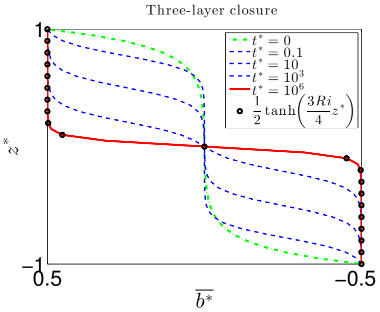
Entrainment across the interface

Mean buoyancy flux

$$\partial_t \overline{b} + \partial_z \overline{J}_b = 0, \quad \overline{J}_b \equiv -l_b e_c^{1/2} \left(\partial_z \overline{b} - \frac{3}{2} Ri \overline{b'^2} \right)$$

Entrainment velocity

$$u_e \equiv \frac{-\overline{J}_b}{\Delta b}$$



Creation of intermediate buoyancy levels within the interface (thickness H/Ri) at a rate $e^{1/2}/\Lambda_b$.

$$u_e = \frac{1}{6} \frac{H}{Ri \Lambda_b} e_c^{1/2}$$

J. Fluid Mech. (1986), *vol.* 166, *pp.* 227–244 227
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On mixing across an interface in stably stratified fluid

By XUEQUAN E† AND E. J. HOPFINGER
Institut de Mécanique, Université de Grenoble et C.N.R.S., Grenoble, France

(Received 19 March 1985 and in revised form 3 December 1985)

- E & Hopfinger 1986 report an interface thickness $\propto Ri^{-1}$ and an entrainment velocity $\propto Ri^{-3/2}$
- The model based on a statistical mechanics approach predicts an tanh shape for the buoyancy profile, with thickness H/Ri, and an entrainment velocity

$$u_e = \frac{1}{6}\frac{H}{Ri\Lambda_b}e_c^{1/2}$$

- Assuming $\Lambda_b \propto Ri^{1/2}$ is consistent with Linden 1973

Summary

- **Equilibrium theory for a Boussinesq fluid**:
  Quantitative predictions for mixing efficiency.
- **Relaxation equations towards the equilibrium state**:
  A reduced 1D model satisfying essential conservation laws of 3D Boussinesq equations.
- **A hierarchy of subgrid-scale models can then be derived, including the effect of forcing and dissipation.**
  Describes a competition between turbulent transport, restratification, and irreversible mixing.
- **We assumed homogeneous isotropic kinetic energy in the domain bulk to study salient features of the model**
  However, anisotropy and inhomogeneity can be taken into account in this framework.

Venaille Gostiaux Sommeria, preprint